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CONTINUAL-DISCRETE MODELLING OF A MULTICOMPONENT LAMINAR BODY BY USING A SYSTEM OF TWO-DIMENSIONAL CONTINUA*

T.A. PRIBYLEVA

A method is considered for constructing continual-discrete models of multicomponent layered bodies by using a system consisting of an arbitrary number of two-dimensional continua with finite intervals between them. Consistency relationships are presented for the fundamental kinematic, deformation, and dynamic parameters which enable rheological relationships to be obtained for the body as a whole taking the properties and nature of the interaction of the individual components into account. An example of the modelling of a thin laminar elastic body is examined. Methods for modelling a biological membrane are discussed.

Physical objects exist for which a direct description is impossible by methods of the mechanics of three-dimensional continuous media, or is insufficiently effective because the physical properties of the object are discrete in one of the directions, i.e., the requirements for the continuity hypothesis /1/ are not satisfied in this direction. The object here posssesses fairly continuous properties in the other two directions and allows of a continual description.

Among the discrete objects in the transverse direction is the shell of a live cell, a biological membrane, say, consisting of several layers of macromolecules where the individual layers include molecules of different species. Moreover, a broad class of laminar and stratified bodies exists, whose properties in the transverse direction can possibly be described by a discrete set of parameters.

In a number of papers (/2-7/, for example) the concept has been introduced of a two-dimensional continuum (a material surface possessing mass) that is characterized by appropriate kinematic, dynamic, and energy parameters. The ideal of modelling multicomponent laminar bodies by using systems of two-dimensional continua /8/ is

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natural*. (*See also: Pribyleva, T.A. Investigations in Biomechanics. Model of a multisheet continuum: kinematics and mass balance. Report 2555, Moscow Univ. Mechanics.Inst., 1981). Certain elements of this approach are elucidated in /9, 10/.

1. Construction of the fundamental continuum and multisheet system.

A two-dimensional continuum (a material surface) is understood to be an infinitely thin physical object satisfying the requirements of the continuity hypothesis /1/ in the longitudinal direction.

The fundamental continuum K is set in correspondence with a three-dimensional body Z as a whole in continual discrete modelling. The geometric surface on which the continuum lies is called the fundamental surface and is denoted by the same symbol K. The surface K (Fig. 1) passes, by definition, through the centre of mass of parts dZ cut out of the body Z by cylindrical elements dv with generators, normal to the surface K. It can be shown that the surface K (given implicitly) exists and is sufficiently smooth under the condition of satisfying the requirements of the continuity hypothesis for the material of the body Z in the longitudinal direction.

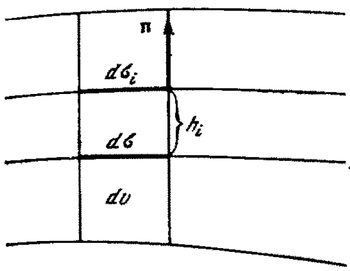

Fig.1

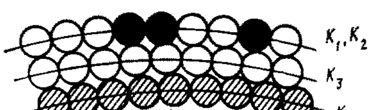

Fig.2

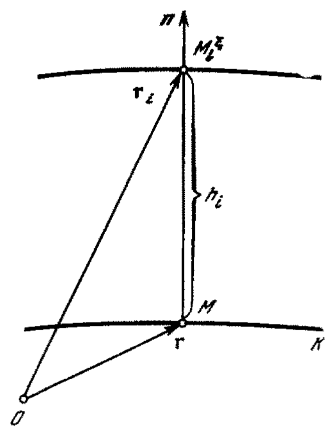

Fig.3

The cylindrical element dv cuts an element $d\sigma$ (Fig.1) out of the continuum K that, by definition, models the part dZ. The appropriate characteristics of the element $d\sigma$ and the part dZ such as the mass, location and velocity of the centre of mass, the total electrical charge, the internal energy, entropy, etc. are here in agreement. The system of forces and moments acting on the element $d\sigma$ should be equivalent to the analogous system of forces and moments acting on the part dZ. Let us assume that the body Z is formed N components Z_i by which we mean, by definition, the individual layers or separate parts of molecules (components) in the layer. For example, the body shown schematically in Fig.2 consists of 4 components.

Denote by dZ_i the part of the components Z_i, cut out by a cylindrical element dv with generatrix normal to the surface K. A multisheet system of two-dimensional continua is constructed on the base of the fundamental surface K as follows: a two-dimensional continuum (sheet) K_i lying on the surface K_i passing through the centre of mass of the parts dZ_i is set in correspondence to each component Z_i. The part $d\sigma_i$ of the continuum K_i included within the cylindrical element dv (Fig.1) models, by definition, the part dZ_i of the component Z_i, where corresponding characteristics of the elements $d\sigma_i$ and the parts dZ_i agree; analogous systems of forces and moments should be equivalent (exactly as in the construction of the fundamental continuum K).

In other words, when constructing the fundamental continuum, the mass of the whole object Z "is removed" to one surface K_i and when constructing the multisheet system the mass of the individual components Z_i (in particular, of the separate layers) is removed to the different surfaces K_i, respectively. A set of elements $d\sigma_i$ (Fig.1) with finite intervals between them is the elementary formation in a multisheet system which is different in principle from the mechanics of three-dimensional mixtures.

We note that separation of a three-dimensional object into layers can be provisional, where the accuracy of the description by using a multisheet system is increased as the number of layers increases.

Some of the continua K_i can lie on one surface and be shifted with respect to another. This enables surface diffusion to be modelled.

A general rule for establishing the correspondence between characteristics of the body Z and the parameters of the fundamental continuum K as well as the components Z_i and the sheets K_i is postulated by introducing the continual discrete description.

The parameters characterizing the properties and interaction of the components Z_i are given initially on different surfaces K_i and are then reduced to a single coordinate system coupled to the surface K.

Underlying the continual-discrete modelling method are consistency relationships that connect the parameters of the continuum K to the parameters of the sheets K_i. Taken into account, in particular, in the derivation of the consistency relationships are the transverse deformations of the layers, the relative longitudinal motion of the layers and their components, and the mass transfer between components and with the external medium.

We emphasize that the balance equations for the continua K and K_i are a trivial extension of the equations presented in /2-7/. The difficulty indeed is to obtain consistency relationships and in constructing systems of equations for a smaller number of variables by using them (as in the mechanics of three-dimensional mixtures also).

A general method is described below for constructing a multisheet system and for establishing a correspondence between the characteristics of the body Z and its components Z_i on the one hand, and the parameters of the continuum K and the sheets K_i on the other; consistency relationships are presented between the kinematic, deformation, and dynamic parameters; an example is given for obtaining the rheological relationships for the fundamental continuum K in the case of a thin elastic laminar body; methods are discussed for constructing models for biological membranes and a rheological equation is presented for the axisymmetric bending of a lipid bilayer when there is mass transfer between the layers.

2. *Kinematic and deformation relationships.*

We introduce Lagrange coordinate systems η^α and η_i^α associated with the centres of masses of the continua K and K_i $(\alpha = 1, 2)$ on the surfaces K and K_i. The centres of mass of the elements $d\sigma$ and $d\sigma_i$ (Fig.1) coincide, respectively, with the centres of mass of the parts dZ and dZ_i. Moreover, we introduce a coordinate system ξ_i^α, "corresponding normally" to the system η^α on the surface K, on the surface K_i. To do this we assume that the point M_i^ξ (Fig.3) lying on the normal to the surface K drawn from the point M with coordinates η^α has the coordinates $\xi_i^\alpha \equiv \eta^\alpha$.

An arbitrary function A_i defined on the surface K_i, is predefined on the surface K according to the rule $A_i(\eta^\alpha) = A_i(\xi_i^\alpha)$, where if the function A_i is computed per unit area of the surface K_i, then an appropriate quantity is introduced per unit area of the surface K:

$$A_i' = k_i A_i, \quad k_i = d\sigma_i/d\sigma \tag{2.1}$$

Therefore, the parameters characterizing the properties of the individual components Z_i and defined initially on the sheets K_i are referred to a single coordinate system η^α on the surface K.

The centre of mass of the set of elements $d\sigma_i$ lies at the point M (Fig.3) so that the following equalities hold

$$\mathbf{r} = \sum_{i=1}^{N} c_i \mathbf{r}_i, \quad c_i = dm_i/dm, \quad dm = \sum_{i=1}^{N} dm_i \tag{2.2}$$

$$\mathbf{r}_i(\eta^\alpha) = \mathbf{r}_i(\xi_i^\alpha) = \mathbf{r}(\eta^\alpha) + h_i(\eta^\alpha)\,\mathbf{n} \tag{2.3}$$

where $\mathbf{r}, \mathbf{r}_i$ are, respectively, the radius-vectors of the surfaces K and K_i (Fig.3), c_i is the mass concentration of the components Z_i, dm and dm_i are, respectively, the masses of the elements $d\sigma$ and $d\sigma_i$ that equal the masses of the parts dZ and dZ_i (see Sect.1), $\mathbf{n}$ is the unit vector normal to the surface K, h_i is the spacing between K_i and K measured along the normal to the surface K, where $h_i \geqslant 0$ if the surface K_i is located on the positive normal to K and $h_i < 0$ otherwise.

In the case of numerical agreement of the values of the coordinate the systems η^α and ξ_i^α have different basis vectors $\mathbf{r}_\alpha$ and $\mathbf{r}_{i\alpha}^\xi$, where the following consistency relationships hold by virtue of (2.3):

$$\begin{aligned} \mathbf{r}_{i\alpha}^\xi &= \mathbf{r}_\alpha + \frac{\partial(h_i\mathbf{n})}{\partial\eta^\alpha} = \mathbf{r}_\alpha - h_i b_\alpha^{\ \beta}\mathbf{r}_\beta + \frac{\partial h_i}{\partial\eta^\alpha}\mathbf{n} \\ a_{i\alpha\beta}^\xi &= \mathbf{r}_{i\alpha}^\xi \cdot \mathbf{r}_{i\beta}^\xi = a_{\alpha\beta} - 2h_i \cdot b_{\alpha\beta} + h_i^2 b_\alpha^{\ \gamma} b_{\beta\gamma} + \frac{\partial h_i}{\partial\eta^\alpha}\frac{\partial h_i}{\partial\eta^\beta} \end{aligned} \tag{2.4}$$

Here $a_{\alpha\beta}$, $b_{\alpha\beta}$ are components of the first and second metric tensors of the surface K and $a_{i\alpha\beta}^\xi$ are components of the first metric tensor of the surface K_i in the system ξ_i^α.

Hence a connection is obtained between the strain tensor $\varepsilon_{i\alpha\beta}^\xi = {}^1/_2\,(a_{i\alpha\beta}^\xi - a_{i\alpha\beta}^{\xi 0})$ for the system ξ_i^α on the sheet K_i and the strain tensor $\varepsilon_{\alpha\beta}$ of the continuum K (here and henceforth the superscript ° denotes the quantity at the initial time t°)

$$\varepsilon^{\xi}_{i\alpha\beta} = \varepsilon_{\alpha\beta} - (h_i b_{\alpha\beta} - h_i^{\circ} \overset{\circ}{b}_{\alpha\beta}) +$$
$$^1/_2 \left(h_i^2 b_\alpha{}^\gamma b_{\beta\gamma} - h_i^{\circ 2} \overset{\circ}{b}_\alpha{}^\gamma \overset{\circ}{b}_{\beta\gamma} + \frac{\partial h_i}{\partial \eta^\alpha} \frac{\partial h_i}{\partial \eta^\beta} - \frac{\partial h_i^\circ}{\partial \eta^\alpha} \frac{\partial h_i^\circ}{\partial \eta^\beta} \right)$$

For the case of a thin body when all the quantities h_i are small compared with the characteristic linear dimensions, this expression takes the simple form /10/

$$\varepsilon^{\xi}_{i\alpha\beta} = \varepsilon_{\alpha\beta} - (h_i b_{\alpha\beta} - h_i^\circ \overset{\circ}{b}_{\alpha\beta}) = \varepsilon_{\alpha\beta} - \chi_i b_{\alpha\beta} - h_i \varkappa_{\alpha\beta} + \chi_i \varkappa_{\alpha\beta} \tag{2.5}$$
$$\chi_i = h_i - h_i^\circ, \quad \varkappa_{\alpha\beta} = b_{\alpha\beta} - \overset{\circ}{b}_{\alpha\beta}$$

Satisfying the identity $\xi_i^\alpha \equiv \eta_i^\alpha$ (meaning that particles of the continua K_i are not shifted relative to the normal to K) for all i during the whole deformation process is the necessary and sufficient condition of the fact that a normal fibre remains rectilinear and normal to the surface K during deformation of the body Z. The tensor $\varepsilon^{\xi}_{i\alpha\beta}$ here coincides identically with the strain tensor $\varepsilon^{\eta}_{i\alpha\beta}$ for the system η_i^α characterizing the deformation of the continuum K_i. Therefore, the tensor $\varepsilon^{\xi}_{i\alpha\beta}$ characterizes the deformation of the sheet K_i under the condition of conservation of a normal fibre.

An expression /9/ holds for the tensor $\varepsilon_{i\alpha\beta}{}^\eta$ in terms of the absolute displacement $\mathbf{u}_i^\xi$ of points of the sheet K_i with constant coordinates ξ_i^α and displacements $\mathbf{U}_i^\circ$ (Fig.4) of the points $M_i^{\circ\eta}$ with the constant coordinates η_i^α relative to the points $M_i^{\circ\xi}$ with the constant coordinates ξ_i^α (under the condition that the points M_i^ξ and M_i^η agree at a given instant t):

$$\varepsilon^{\eta}_{i\alpha\beta} = \varepsilon^{\xi}_{i\alpha\beta} + \varepsilon^{U}_{i\alpha\beta} - \frac{1}{2}\left(\frac{\partial \mathbf{u}_i^\xi}{\partial \eta_i^\alpha} \frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\beta} + \frac{\partial \mathbf{u}_i^\xi}{\partial \eta_i^\beta} \frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\alpha} \right)$$
$$\varepsilon^{U}_{i\alpha\beta} = \left(\frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\alpha} \mathbf{r}^{\xi}_{i\beta} + \frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\beta} \mathbf{r}^{\xi}_{i\alpha} - \frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\alpha} \frac{\partial \mathbf{U}_i^\circ}{\partial \eta_i^\beta} \right)$$

The absolute displacements of particles of the continuum K_i (Fig.4) are

$$\mathbf{u}_i^\eta = \mathbf{U}_i^\circ + \mathbf{u}_i^\xi$$

The quantities $\mathbf{U}_i^\circ$ characterize the deviation of the "prototype" $C^\circ D^\circ$ of the fibre CD normal to the surface K at a given instant from the normal $\mathbf{n}^\circ$ to the initial position K° of the surface K. For deformations with a normal fibre conserved $\mathbf{U}_i^\circ \equiv 0$.

Let us introduce the absolute velocities of the continua K and K_i (that coincide with the velocities of the centres of mass of the parts dZ and dZ_i) as the velocities of points with the constant coordinates η^α and η_i^α respectively. The relative motion of the continua K_i and K reflecting the motion of the components Z_i relative to the common centre of mass is characterized by the velocity $\mathbf{w}_i^\eta = \mathbf{v}_i - \mathbf{v}$, where

$$\mathbf{w}_i^\eta = \mathbf{W}_i + \mathbf{w}_i^\xi, \quad \mathbf{w}_i^\xi = \mathbf{v}_i^\xi - \mathbf{v}, \quad \mathbf{W}_i = \mathbf{v}_i - \mathbf{v}_i^\xi \tag{2.6}$$

and $\mathbf{v}_i^\xi$ is the absolute velocity of the points M_i^ξ with the constant coordinates ξ_i^α; $\mathbf{W}_i$ is the velocity of motion of the particles of the continuum K_i along the surface K_i (relative to the point M^ξ_i, that coincides with the particle at this time). The velocity $\mathbf{w}_i^\xi$ of motion of the point M_i^ξ relative to the point M can be expressed in terms of the parameter of the continuum K and the quantity h_i.

The consistency relationships for the strain rate tensor components of the continua K and K_i are obtained from (2.6). The connections between the derivatives with respect to the coordinates on the surfaces K and K_i that are derived from (2.4) for the basis vectors, are used here.

Because there are finite intervals (Fig.1) between the elements $d\sigma_i$ the velocity $\mathbf{v}$ of the common centre of mass lying on the element $d\sigma$ is composed of the average mass velocity of the component motion and the velocity $\mathbf{w}$ of the displacement of the centre of mass because of the change in the mass concentrations c_i of the components Z_i

$$\mathbf{v} = \sum c_i \mathbf{v}_i + \mathbf{w} \tag{2.7}$$
$$\mathbf{w} = \sum h_i \frac{dc_i}{dt} \mathbf{n} = -\sum c_i \frac{dh_i}{dt} \mathbf{n} = -\sum c_i \mathbf{w}_i^\xi$$

where $d/dt = \partial/\partial t\,|_{\eta\alpha}$ is the individual derivative with respect to time for the continuum K. Here and everywhere henceforth, the summation is over the subscript i.

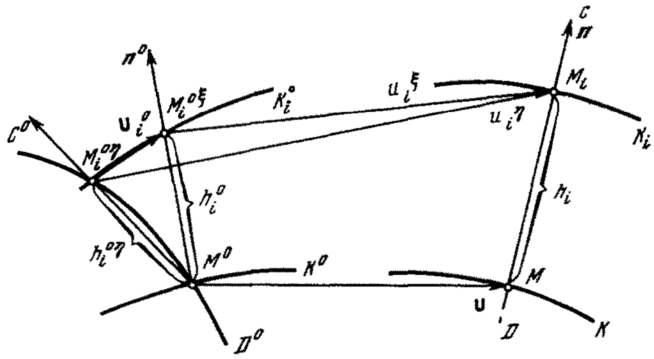

Fig.4

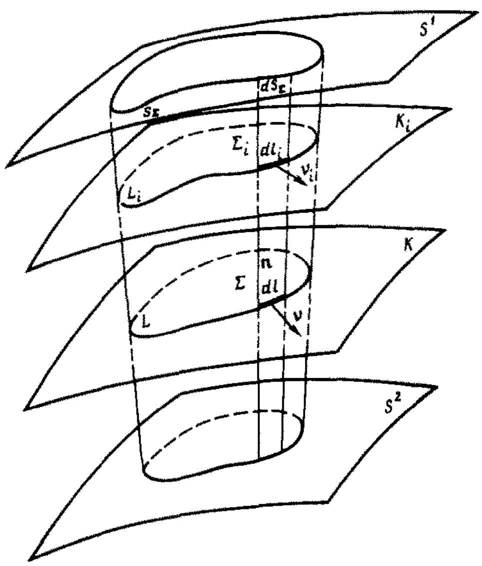

Fig.5

There is no such effect in the three-dimensional mechanics of mixtures because the components are not separated in space. If it is assumed that the displacements of the common centre of mass are independent because of the motion of the components and because of the change in concentrations, (2.7) can be obtained from (2.2) for $\mathbf{r}_i = \mathrm{const}$.

The change in concentrations c_i is subject to the diffusion equations

$$\rho dc_i/dt = -k_i \operatorname{div}_i \mu_i + m_i^{e'} - c_i m^e \tag{2.8}$$

where $\rho = dm/d\sigma$ is the surface density of the continuum K, $m_i^{e'}$ is the mass influx to the continuum K_i per unit time per unit area of the surface K (see (2.1)) $m^e = \sum m_i^{e'}$ is the mass influx to the continuum K $\mu_i = \rho_i \mathbf{W}_i$ is the mass flux vector along the surface K_i ($\rho_i = dm_i/d\sigma_i$ is the surface density of the continuum K_i), and div_i is the surface divergence operator on the sheet K_i. Eq.(2.8) is obtained from the equations of continuity for the continua K and K_i and the expressions for the derivatives of the quantity k_i with respect to time /9/.

3. *Dynamic relationships.* Let us construct the surface S_Σ (Fig.5) by using the normals to the surface K restored from the boundary L of a certain domain Σ on the surface K. The surface S_Σ cuts parts Z_Σ and $Z_{\Sigma i}$ respectively from a three dimensional body Z and the components Z_i and extracts a domain Σ_i, bounded by the contour L_i on the surface K_i. An elementary strip dS_Σ cutting the element $dl_{i\nu}$ out of the contour L_i here

corresponds to an element dl_ν (Fig.5) with a unit tangential normal $\boldsymbol{\nu}$.

By virtue of the definitions in Sect.1 the systems of forces and moments acting on the parts Z_Σ and $Z_{\Sigma i}$, are equivalent to analogous systems of forces and moments acting on the respective parts of the continua K and K_i included within the domains Σ and Σ_i.

We examine the forces and moments acting on the parts Z_Σ and $Z_{\Sigma i}$ from the rest of the whole body Z and distributed over the side surface S_Σ. We reduce the appropriate systems of forces and moments arriving at the elementary strip dS_Σ to the principal forces $d\mathbf{p}_{z\nu}$ and $d\mathbf{p}_{zi\nu}$ applied, respectively, to the elements dl_ν and $dl_{i\nu}$ and to the moments $d\mathbf{M}_{z\nu}$ and $d\mathbf{M}_{zi\nu}$. We extract the additional moments $d\mathbf{M}_{z\nu p}$ and $d\mathbf{M}_{zi\nu p}$ that occur because of removal of the forces distributed along the strip dS_Σ to the elements dl and dl_i: respectively

$$d\mathbf{M}_{z\nu} = d\mathbf{M}_{z\nu 0} + d\mathbf{M}_{z\nu p}, \quad d\mathbf{M}_{zi\nu} = d\mathbf{M}_{zi\nu 0} + d\mathbf{M}_{zi\nu p} \tag{3.1}$$

In order to satisfy the requirement of equivalence of the systems of forces and moments we should set

$$\mathbf{p}_\nu\, dl_\nu = d\mathbf{p}_{z\nu}, \quad \mathbf{M}_\nu\, dl_\nu = d\mathbf{M}_{z\nu}, \quad \mathbf{p}_{i\nu}\, dl_{i\nu} = d\mathbf{p}_{zi\nu}, \quad \mathbf{M}_{i\nu}\, dl_{i\nu} = d\mathbf{M}_{zi\nu}$$

where $\mathbf{p}_\nu, \mathbf{p}_{i\nu}$ and $\mathbf{M}, \mathbf{M}_{i\nu}$ are, respectively, the densities of the linear forces and moments for the continua K and K_i. We note that by virtue of the definitions of the quantities $d\mathbf{p}_{zi\nu}$ and $d\mathbf{M}_{zi\nu}$ the quantities $\mathbf{p}_{i\nu}$ and $\mathbf{M}_{i\nu}$ reflect not only the stress in the sheet K_i but also the appropriate forces and moments from the remaining continua. When there are no cross-forces and interaction moments of the components distributed over the surface S_Σ, then $\mathbf{p}_{i\nu}$ and $\mathbf{M}_{i\nu}$ are the stress in the continuum K_i.

In conformity with (3.1), the densities $\mathbf{M}_{\nu p}$ and $\mathbf{M}_{i\nu p}$ of the additional moments

$$\mathbf{M}_\nu = \mathbf{M}_{\nu 0} + \mathbf{M}_{\nu p}, \quad \mathbf{M}_{i\nu} = \mathbf{M}_{i\nu 0} + \mathbf{M}_{i\nu p}$$

are extracted.

It can be shown by using the analysis of the partition of the strip dS_Σ by planes perpendicular to $\mathbf{n}$ that

$$\mathbf{M}_{\nu p} = \sum \lambda_{i\nu}\mathbf{M}_{i\nu p} + \sum h_i \mathbf{n} \times \lambda_{i\nu}\mathbf{p}_{i\nu}, \quad \lambda_{i\nu} = dl_{i\nu}/dl_\nu \tag{3.2}$$

from which by using the equalities

$$d\mathbf{p}_{z\nu} = \sum d\mathbf{p}_{zi\nu}, \quad d\mathbf{M}_{z\nu 0} = \sum_i d\mathbf{M}_{zi\nu 0}$$

we obtain the relationships

$$\begin{gathered}\mathbf{p}_\nu = \sum \lambda_{i\nu}\mathbf{p}_{i\nu}, \quad \mathbf{M}_{\nu 0} = \sum \lambda_{i\nu}\mathbf{M}_{i\nu 0} \\ \mathbf{M}_\nu = \sum \lambda_{i\nu}\mathbf{M}_{i\nu} + \sum h_i \mathbf{n} \times \lambda_{i\nu}\mathbf{p}_{i\nu}\end{gathered} \tag{3.3}$$

The components of the stress tensors $p^{\beta\alpha}$, $p_i^{\beta\alpha}$ and of the moments $M^{\beta\alpha}$, $M_i^{\beta\alpha}$ and of transverse stress vector $p^{3\alpha}$, $p_i^{3\alpha}$ and of the moments $M^{3\alpha}$, $M_i^{3\alpha}$ are determined, respectively, for the continua K and K_i from the expansions

$$\begin{gathered}\mathbf{p}^\alpha = p^{\beta\alpha}\mathbf{r}_\beta + p^{3\alpha}\mathbf{n}, \quad \mathbf{M}^\alpha = M^{\beta\alpha}\mathbf{r}_\beta + M^{3\alpha}\mathbf{n} \\ \mathbf{p}_i^\alpha = p_i^{\beta\alpha}\mathbf{r}_{i\beta} + p_i^{3\alpha}\mathbf{n}_i, \quad \mathbf{M}_i^\alpha = M_i^{\beta\alpha}\mathbf{r}_{i\beta} + M_i^{3\alpha}\mathbf{n}_i\end{gathered}$$

Here $\mathbf{n}_i$ is the unit normal vector to the surface K_i, $\mathbf{p}^\alpha$, $\mathbf{p}_i^\alpha$ and $\mathbf{M}^\alpha$, $\mathbf{M}_i^\alpha$ are, respectively, the stresses and moments on length elements with normals $\mathbf{r}^\alpha$ and $\mathbf{r}_i^\alpha$ (vectors of the mutual basis), where the following hold:

$$\mathbf{p}_\nu = \mathbf{p}^\alpha\nu_\alpha, \quad \mathbf{M}_\nu = \mathbf{M}^\alpha\nu_\alpha, \quad \boldsymbol{\nu} = \nu_\alpha\mathbf{r}^\alpha$$

The relationships

$$\begin{gathered}p^{\beta\alpha} = \sum k_i\,(p_i^{\beta\alpha} - h_i b_\gamma^{\ \beta} p_i^{\gamma\alpha}), \quad p^{3\alpha} = \sum_i k_i p_i^{3\alpha} \\ M^{\beta\alpha} = \sum_i k_i\,[M_i^{\beta\alpha} - h_i b_\gamma^{\ \beta} M_i^{\gamma\alpha} + h_i \partial_{\delta}^{\cdot\beta}\,(p_i^{\delta\alpha} - h_i b_\gamma^{\ \delta} p_i^{\gamma\alpha})] \\ M^{3\alpha} = \sum_i k_i M_i^{3\alpha} \\ (\partial_{\cdot 1}^{1\cdot} = -\sqrt{a}\, a^{12}, \quad \partial_{\cdot 2}^{1\cdot} = \sqrt{a}\, a^{11}, \quad \partial_{\cdot 1}^{2\cdot} = -\sqrt{a}\, a^{22}, \quad \partial_{\cdot 2}^{2\cdot} = \sqrt{a}\, a^{12})\end{gathered} \tag{3.4}$$

resulting from (3.3) were obtained /10/ under the condition that the quantities h_i do not change along the surface K (it can here be shown that $\mathbf{n}_i = \mathbf{n}$).

Consistency relationships between the remaining dynamic parameters (mass forces and moments, etc.) were examined in /9/ and, in particular, relationships are obtained for appropriate additional moments that have a form analogous to (3.2).

Because of the existence of additional moments associated with removal of the forces on the surfaces K and K_i, the stress tensors $p^{\alpha\beta}$ and $p_i^{\alpha\beta}$ turn out to be non-symmetric for a broad class of cases, in contrast to the three-dimensional mechanics of continuous media.

4. On obtaining the closing relationships. The equations of motion and moments for the continuum K have the form

$$\rho\, d\mathbf{v}/dt = \nabla_\alpha \mathbf{p}^\alpha + \boldsymbol{\Phi} + m^e(\mathbf{v} - \mathbf{v}_M) \qquad (4.1)$$
$$\rho\, d\mathbf{k}/dt = \mathbf{r}_\alpha \times \mathbf{p}^\alpha + \nabla_\alpha \mathbf{M}^\alpha + \mathbf{N} + (\mathbf{k}_m - \mathbf{k})\, m^e$$

where ∇_α is the surface covariant differentiation operator, $\boldsymbol{\Phi}$ and $\mathbf{N}$ are the surface densities of the external forces and moments, $\mathbf{k}$ is the density of the internal kinetic moment, and $\mathbf{v}_M$ and $\mathbf{k}_M$ are the average-mass velocity and the density of the internal kinetic moment of particles flowing in from outside. Appropriate equations for the continuum K_i are analogous in form.

The consistency relationships presented in Sects.2 and 3 enable rheological relations to be obtained for the continuum K modelling the object Z as a whole on the basis of the rheological relationships for the continua K_i modelling the individual components Z_i. For instance, deformation with conservation of the normal fibre of a thin elastic layered body was examined in /10/. It was assumed here that the individual layers do not resist bending deformations and can be simulated by using linearly elastic two-dimensional continua

$$p_i^{\beta\alpha} = A_i^{\beta\alpha\gamma\delta} \varepsilon_{i\gamma\delta} + p_i^{\circ\beta\alpha}, \quad \mathbf{M}_i^\alpha \equiv 0 \qquad (4.2)$$

where $A_i^{\beta\alpha\gamma\delta}$ are the surface elasticity coefficients, and $p_i^{\circ\beta\alpha}$ are initial stress tensor components in the sheets K_i, $\varepsilon_{i\gamma\delta} \equiv \varepsilon_{i\gamma\delta}^{\xi} \equiv \varepsilon_{i\gamma\delta}^{\eta}$.

These equalities are the closing relationships for the equilibrium equations for the sheets K_i.

Rheological relationships were obtained for the continuum K from (4.2) by using the consistency relationships (2.5) and (3.4)

$$\begin{gathered} p^{\beta\alpha} = \sum D_i^{\beta\alpha\gamma\sigma} (\varepsilon_{\gamma\sigma} - \chi_i b_{\gamma\sigma} - h_i \varkappa_{\gamma\sigma} + \chi_i \varkappa_{\gamma\sigma}) + p^{\circ\beta\alpha *}, \quad p^{3\alpha} = 0 \\ M^{\beta\alpha} = \sum h_i \partial_{\lambda\cdot}^{\cdot\beta} D_i^{\lambda\alpha\gamma\sigma} (\varepsilon_{\gamma\sigma} - \chi_i b_{\gamma\sigma} - h_i \varkappa_{\gamma\sigma} + \chi_i \varkappa_{\gamma\sigma}) + M^{\circ\beta\alpha *}, \quad M^{3\alpha} = 0 \\ p^{\circ\beta\alpha *} = \sum k_i (p_i^{\circ\beta\alpha} - h_i b_\gamma{}^\beta p_i^{\circ\gamma\alpha}), \quad M^{\circ\beta\alpha *} = \sum k_i (M_i^{\circ\beta\alpha} - h_i b_\gamma{}^\beta M_i^{\circ\gamma\alpha}), \\ D_i^{\beta\alpha\gamma\sigma} = k_i (A_i^{\beta\alpha\gamma\sigma} - h_i b_\delta{}^\beta A_i^{\delta\alpha\gamma\sigma}) \end{gathered} \qquad (4.3)$$

where the last three equalities determine, respectively, the total initial stresses and moments in the sheets K_i and the effective elasticity coefficients. These relationships include the dependence of the stress and moments on the bending deformations $\varkappa_{\gamma\delta}$ (unlike the relationships (4.2) for the sheet K_i) and enable the resistance to bending of a laminar body because of tension and compression of the individual layers when there is no resistance to bending of the layers, to be modelled.

The equalities (4.3) are the closing relationships for Eqs.(4.1) represented in coordinate form for $\mathbf{v} \equiv \mathbf{v}_M \equiv 0$, $\boldsymbol{\Phi} \equiv 0$, $\mathbf{N} \equiv 0$, $\mathbf{k} \equiv \mathbf{k}_M \equiv 0$. Utilization of the consistency relationships enables the stress and strain of the individual sheets K_i to be eliminated and enables the rheological constants of the continua K_i to be taken into account. The relationships for the quantities χ_i characterizing the transverse strains of the object Z should be obtained from additional considerations, for instance, from the condition of volume incompressibility or an analysis of the interaction forces between the layers. It is possible to formulate problems in which $\chi_i \equiv 0$, which means conservation of the normal fibre length (as in the classical theory of shells).

Interaction forces between the continua K_i will enter into the expanded model. The necessary additional relationships for these quantities should be obtained from an analysis of the interactions between the components Z_i.

5. On methods of modelling a biological membrane. Among the various kinds of biological membranes, the erythrocyte membrane is the most prevalent object of mechanical investigations. Depending on its mechanical properties are, for instance, blood viscosity, the ability for erythrocytes to pass through capillaries, the allowable operation time when using artifical blood circulation apparatus, etc. /11, 12/.

The structural basis of the majority of biological membranes, and in particular the erythrocyte membrane (Fig.6), where 1 are proteins, 2 are lipids, 3 are protein cytoskeletons, and 4 are lycocalyx) is a bilayer consisting of lipid (fat) molecules, where the individual monolayers can be exchanged by molecules and have different mechanical properties. The liquid-crystal properties of the lipid bilayer permit parts of the included protein molecules to be displaced (diffused) in the plane of the membrane. The membrane thickness can change, for instance, under the action of Coulomb forces.

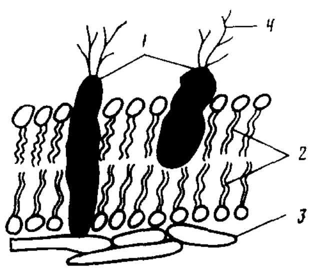

Fig.6

A multisheet system modelling a biological membrane can, depending on the specific situation, consist of a different number of sheets. Thus, when studying the surface diffusion of proteins it is sufficient to differentiate two two-dimensional continua lying on one surface and corresponding to lipid and protein components. Such problems were solved for multicomponent interfacial surfaces (/5/, say).

To obtain the rheological relationships for the membrane as a whole, it is visibly meaningful to consider a system of four sheets separated by finite spacings and corresponding to two lipid monolayers (with the inclusion of appropriate protein components), an internal cytoskeleton and a glycocalyx. The first two sheets are two-dimensional viscous fluids and the other two are two-dimensional viscoelastic continua.

Within the framework of this examination, electrical phenomena in the membrane, mass transfer between layers, etc., as well as their connection with the mechanical behaviour of the membrane can be taken into account.

The occurrence of effective viscoelasticity of the bilayer under bending due to mass trasfer directed at equilibration of the densities ρ_i of the monolayers is an example. In the axisymmetric case the rheological equation that connects the bending moment M and the bending defromation $\varkappa$ of the lipid bilayer, has the form

$$dM/dt + 2h^2A d\varkappa/dt + 2\gamma M + 4q\gamma h^2 A\varkappa = 0 \tag{5.1}$$

where h and A are, respectively, the thickness and surface elasticity coefficient of the individual monolayer (for simplicity the layer properties are assumed identical), and γ and q are mass transfer parameters whose law is taken in the form

$$d\,(dm_1)/dt = \gamma\,(\rho_2 - \rho_1 - q\rho h/R)\,d\sigma$$

Here R is the radius of the surface K that is a part of a cylindrical surface by assumption.

Eq.(5.1) is obtained from the relations (2.5), (3.4), (4.2) and (4.3) under the assumption that the layers are identical and do not slip relative to each other.

The characteristic relaxation times in (5.1) are determined completely by the mass transfer parameters γ and q.

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CONDITIONS ON SURFACES OF DISCONTINUITY IN A RIGID-PLASTIC ANALYSIS*

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Relationships are established on the surfaces of discontinuity for a rigid plastic analysis of inhomogeneous bodies, particularly bodies with piecewise-continuous properties. They are derived as necessary conditions for the static and dynamic load coefficients to be equal. In the special case of homogeneous bodies they are identical with the well-known Hill conditions; the necessity of the latter is thereby established. The connection of the different formulations of extremal problems of limit load theory is discussed in deriving the relationships. The mechanical meaning of the relationships obtained and certain of their properties are examined.

1. Formulation of the problem. Conditions on surfaces of discontinuity in a rigid plastic analysis are examined in this paper from the viewpoint of limit load theory.

The problem of the limit load theory. Let a body occupy a domain Ω and be subjected to mass forces with density $\mathbf{f}$ and a surface load with density $\mathbf{q}$ applied to a part S_q of the body surface. There is a set C_x of allowable stresses for each point x of the body. The stress field that is an inner point of the set of allowable stress fields is called safe. The main question of limit load theory is to clarify whether or not it is possible to equilibrate a given load $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ and a safe stress field.

The static extremal problem. A field σ of allowable stresses is called statically allowable for a load $m\mathbf{l}$, $m \geqslant 0$, $\mathbf{l} = (\mathbf{f}, \mathbf{q})$ if it equilibrates this load; in this case the number $m_s(\sigma) = m$ is called the static coefficient of the load $\mathbf{l}$. The exact upper bound $\alpha_l = \sup m_s(\sigma)$ is called the static limit coefficient of the load $\mathbf{l}$. The load $m\mathbf{l}$ can be equilibrated to a safe stress field for $0 \leqslant m < \alpha_l$ and it is impossible to equilibrate thus for $m \geqslant \alpha_l$ /1/. Therefore, to answer the fundamental question of limit load theory it is necessary to find or estimate the quantity α_l, that is called the safety factor of the load $\mathbf{l}$ also in connection with the assertion presented.

Kinematic extremal problems. Kinematic extremal problems that are formulated in the following manner play an important part in finding the load safety factor. A dissipation function /2-6/ is associated with a set of allowable stresses C_x ($\mathbf{e}$ is a symmetric tensor of the second rank)

$$d(x; \mathbf{e}) = d_x(\mathbf{e}) = \sup\{\sigma_* \cdot \mathbf{e}: \sigma_* \in C_x\} \tag{1.1}$$

The strain rate $\mathbf{e}(\mathbf{u})$ and dissipation

$$\int_\Omega d_x(\mathbf{e}(\mathbf{u}))\, dx$$

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